Lectures on Challenging Mathematics

Algebra 1

Part 1

Summer 2017

Edited by

Zuming Feng Yunhua Xu Chengde Feng Ivan Borsenco

Contents

1.7 Multiplication of numbers and variables (part 2)

1. The distributive property states that $(-1)x + 1x$ is the same as $(-1+1)x$, and this is 0. It follows that $(-1)x$ is the same as $-x$. Explain why. Then use similar reasoning to explain why $(-x)y$ is the same as $-(xy)$. By the way, is it correct to say, " $-x$ is a negative number"?

2. In a *term* that is the product of a number and a variable, the number is called the *coefficient* of the variable. The distributive property allows you to combine like terms that have the same variable by adding coefficients. For instance, $5x$ and $(-3x)$ are like terms, but $5x$ and -3 are not like terms. An expression is *simplified* if it has no grouping symbols like parenthesis and if all the like terms have been combined. Simplify the expression by combining like terms.

   (a) $8x + 3x$ (b) $4x + 2 - 2x$ (c) $3 - 2(4 + x)$ (d) $(5 - 2x)(-4) + x$

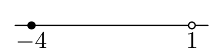
3. For each of the following equations, write the *left-hand side* of the equation as the product of x and another quantity and solve the equation for x:

   (a) $16x + 7x = 46$ (b) $12x - 6x = 3$ (c) $ax + bx = 10$ (d) $px - qx = r$

4. A *sequence* is a list of numbers (or terms). An *arithmetic progression* or *arithmetic sequence* is a list in which each term is obtained by adding a constant number to the preceding term. This constant number is known as the *common difference*. Consider the arithmetic sequence of numbers $2, 5, 8, 11, 14, \ldots$, in which each number is three more than its predecessor.

   (a) Find the next three numbers in the sequence.
   (b) Find the 100^{th} number in the sequence.
   (c) Using the variable n to represent the position of a number in the sequence, write an expression that allows you to calculate the n^{th} number. The 200^{th} number in the sequence is 599. Verify that your expression works by evaluating it with n equal to 200.

5. The diagram on the right illustrates those numbers that are greater than or equal to -4 and less than 1.
   Mathematically we can represent them as numbers satisfying inequalities $-4 \leq x < 1$.
   Use mathematical notation to represent the intervals described below.



   (a) All numbers that are greater than 1 or less than -3.
   (b) All numbers that are greater than -5 and less than or equal to 4.
   (c) All numbers such that twice of that number is greater than or equal to 1.
   (d) All numbers such that a third of that number is less than 3 and greater than -2.
   (e) All numbers whose absolute value is less than 3.
   (f) All numbers whose square is greater than or equal to 4.

1.9 Percentage (part 1)

1. At the Small Fry Cafe a sales tax is added to the bill, equal to 5% of the amount spent on food. After dinner Corey is debating whether or not to consume a jumbo banana split, which cost \$4.60. If Corey decides to order the banana split, then by how much will the amount of tax Corey pays increase?

2. If M is 30% of Q, Q is 20% of P, and N is 50% of P, compute M/N.

3. Boston Red Sox started their season with 15 wins and 10 losses. In order to make the playoff, the team's winning percentage of the 162-game season needs to be at least 70%. To the nearest hundredth of a percent, what should be the lowest winning percentage of the rest of the season for which it is possible for the team to make the playoff?

4. At Jefferson Summer Camp, 60% of the children play soccer, 30% of the children swim, and 40% of the soccer players swim. To the nearest whole percent, what percent of the non-swimmers play soccer?

5. Randy has 25% more money than Sandy, and 20% more money than Mandy, who has \$1800. How much money does Sandy have?

2.5 Rules for operations with integer exponents (part 2)

1. The diameter of an atom is so small that it would take about 10^8 of them, arranged in a line, to span one centimeter. It is thus a plausible estimate that a cubic centimeter contains about $10^8 \times 10^8 \times 10^8 = (10^8)^3$ atoms. Write this huge number as a power of 10.

2. Replace the triangles in

$$\frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x} = x^{\Delta} \quad \text{and} \quad \frac{6^9}{6^4} = 6^{\nabla}$$

by correct exponents.

3. Faced with the problem of dividing 5^{24} by 5^8, Brook is having trouble deciding which of these four answers is correct: 5^{16} or 5^3. Your help is needed. Once you have answered Brook's question, experiment with other examples of this type until you are ready to formulate the *common-base principle for division* that tells how to divide b^m by b^n and get another power of b. Then apply this principle to the following situations:

   (a) The Earth's human population is roughly 7×10^9, and its total land area, excluding the polar caps, is roughly 5×10^7 square miles. If the human population were distributed uniformly over all available land, approximately how many persons would be found per square mile?

   (b) At the speed of light, which is 3×10^8 meters per second, how many seconds does it take for the Sun's light to travel the 1.5×10^{11} meters to the Earth?

4. In each of the following, find the correct value for ∇:

   (a) $y^4 y^7 = y^{\nabla}$ (b) $y^{12} y^{\nabla} = y^{36}$ (c) $y^{\nabla} = \dfrac{y^{11} y^{13}}{y^{17}}$ (d) $\left(y^{\nabla}\right)^3 = y^{27}$

5. Simplify the expressions:

   (a) $2^{15} \cdot 2^{15} \cdot 2^{15} \cdot 2^{15}$ (b) $2^{15} + 2^{15} + 2^{15} + 2^{15}$

   (c) 25^5 (d) $4^4 \cdot 6^6 \cdot 9^9$

   (e) $\dfrac{12^{18}}{18^{12}}$ (f) $\dfrac{(2^3)^4}{(4^3)^2}$

2.8 Linear equations and word problems (part 2)

1. Solve the following equations and check your result by plugging it back in:

   (a) $9(a-4) = 5(3a-2)$

   (b) $\frac{2}{3}(9x+6) = 13-(1-2x)$

   (c) $\frac{x}{5} = \frac{x}{7} + \frac{x}{11}$

   (d) $\frac{|x|}{3} = \frac{2|x|}{5} + \frac{4}{7}$

2. It took Lara five days to read a novel. Each day after the first day, Lara read half as many pages as the day before. If the novel was 248 pages long, how many pages did she read on the first day?

3. A gazelle can run 73 feet per second for several minutes. A cheetah can run faster (88 feet per second) but can only sustain its top speed for about 20 seconds before it is worn out. How far away from the cheetah does the gazelle need to stay for it to be safe?

4. Sammy felt very generous. He gave Becky half his pennies. Then, an hour later, Sammy gave Janie one fourth of his remaining pennies. Shortly after, Mary borrowed half the pennies that Sammy had left. Sammy then had 12 pennies. How many pennies did Sammy give to Becky?

5. Three water pipes are used to fill a swimming pool. Used alone, the first pipe takes 8 hours, the second pipe takes 12 hours, and the third pipe takes 24 hours. If all three pipes are used together, how many hours will take to fill the pool?

3.6 Solving for the right variable (part 2)

1. Solve for x:
   (a) $3x - 4 = 11$ (b) $-2x + 5 = -1$ (c) $2(x - 3) = 4$ (d) $-3(2x + 1) = 5$
   (e) $ax + b = c$ (f) $ax + bx = c$ (g) $a(bx + c) = d$ (h) $a(bx + c) = dx$

2. Day student Avery just bought 10 gallons of gasoline, the amount of fuel used for the last 355 miles of driving. Being of a curious sort, Avery wondered how much fuel had been used in city driving (which takes one gallon for every 25 miles) and how much had been used in freeway driving (which takes one gallon for each 40 miles). Avery started by guessing 6 gallons for the city driving, then completed the first row of the guess-and-check table below. Notice the failed check. Make your own guess and use it to fill in the next row of the table.

City, g	Freeway, g	City, mi	Freeway, mi	Total, mi	Target	Check
6	$10 - 6 = 4$	$6(25) = 150$	$4(40) = 160$	$150 + 160 = 310$	355	no

3. A rectangular box has dimensions $a \times b \times c$.
   (a) Find a formula in terms of a, b, c for the volume of this box.
   (b) Evaluate this formula when $a = 2$ cm, $b = 2.5$ cm, $c = 3$ cm.
   (c) Write a formula for the total surface area of the box.
   (d) Evaluate this formula when $a = 2$ cm, $b = 2.5$ cm, $c = 3$ cm.

4. The volume of a circular cylinder with radius r and height h is given by the formula $V = \pi r^2 h$. Can you explain why?
   (a) To the nearest tenth of a cubic cm, find the volume of a cylinder that has a 15 cm radius and is 12 cm high.
   (b) Solve the volume formula for h. If the volume of a cylinder is 1000 cm^3 and the radius is 10 cm, find h to the nearest tenth of a cm.

5. Exponents are routinely encountered in science, where they help to deal with small numbers. For example, the diameter of a proton is 0.0000000000003 cm. Explain why it is logical to express this number in scientific notation as 3×10^{-13}. The surface area of a sphere is equal to $4\pi R^2$ and the volume of a sphere is equal to $\frac{4}{3}\pi R^3$, where R is the radius of a sphere. Calculate the surface area and the volume of a proton. Express your answers in scientific notation.

3.9 Graphing linear equations

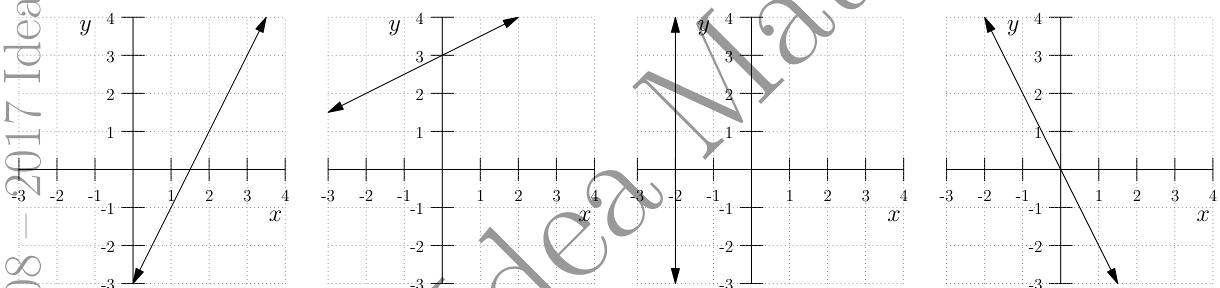
1. A *solution of an equation* in two variables x and y is an ordered pair (x, y) that makes the equation true. The *graph of an equation* in x and y is the collection of all points (x, y) that are solutions of the equation.

   For each equation below, write a table with four different pairs (x, y) that are solutions to the equation. Use these tables to graph both equations:

   (a) $y + 2 = 3x$ (b) $y = -\frac{1}{2}x + 4$

2. Match the equation with its graph:

   (a) $x = -2$ (b) $2x - y = 3$ (c) $6x + 3y = 0$ (d) $-x + 2y = 6$



3. A *linear equation* in x and y is an equation that can be written in the form $ax + by = c$. In the case, when y is constant while x varies, we set $a = 0$. Similarly, if y can assume any value when x varies, we set $b = 0$.

   Graph each of the following equations:

   (a) $x + y = 0$ (b) $x = -3$
   (c) $y = 2$ (d) $2x - y - 1 = 0$

4. Find coordinates for the points where the line $3x + 2y = 12$ intersects the x-axis and the y-axis. These points are called the *x-intercept* and *y-intercept*, respectively.

   Find the x-intercept and the y-intercept of the given equations. Use these points to make a quick sketch of the line.

   (a) $2x + 3y = 6$ (b) $4x - 5y = -35$

5. For of each of the following line equations find the x-intercept and the y-intercept and graph the line:

   (a) $3.5x + 7y = 14$ (b) $y = 2x - 4$

4.5 Point-slope form

1. Plot points $A = (1, 1)$, $B = (4, 5)$, and $C = (9, 11)$. It looks like these points lie on the same line! Do you agree or not? Provide mathematical evidence for your answer.

2. Let $P = (x, y)$ and $Q = (1, 5)$. Write an equation that states that the slope of line PQ is 3. Show how this slope equation can be rewritten in the form $y - 5 = 3(x - 1)$. This is the *point-slope form* of the linear equation. Explain the terminology. Find coordinates for three different points P that fit this equation.

3. (Continuation) What do the lines $y = 3(x - 1) + 5$, $y = 2(x - 1) + 5$, and $y = -\frac{1}{2}(x - 1) + 5$ all have in common? How do they differ from each other?

4. A line passing through the points $(-3, -2)$ and $(5, 6)$. Determine if the point $(10, 12)$ lies on the line. Explain your reasoning. Find y so that the point $(7, y)$ is on the line.

5. A dining hall provides the following monthly meal plans:

   - Option A: \$240 for the first 30 meals and \$10 for each additional meal.
   - Option B: \$420 for the first 60 meals and \$10 for each additional meal.
   - Option C: \$540 per month, all meals are included.

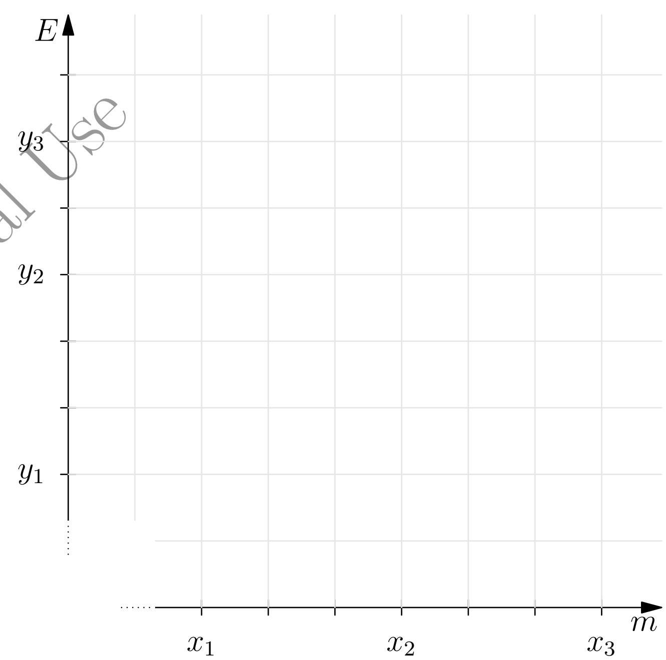
   This month John is going to have m meals in the dining hall. Let E denote the total cost per month he is going to pay for his meals. We are going to compare meal plans by graphs.

   (a) For each plan express E in terms of m.

   (b) In order to graph the equations of costs, what should be the reasonable values of x_1, x_2, x_3 and y_1, y_2, y_3 in the coordinate system below? That is we are trying to find a reasonable *scale* for each of x and y axis.

   (c) Based on your choice of the scale, graph the cost of each meal.

   (d) Compare the benefits of each plan.



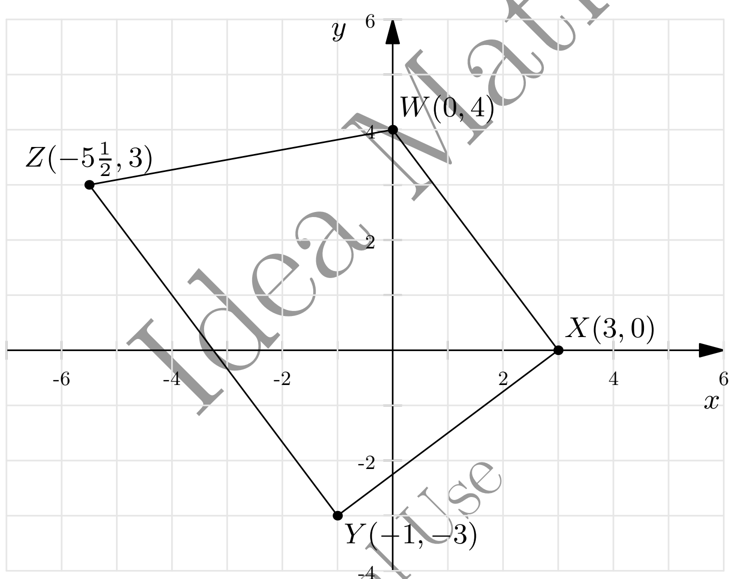
4.8 The slope of a line (part 3)

1. Find the equations of at least three lines that intersect each other at the point $(6, -2)$.

2. Find values for a and b that make $ax + by = 14$ parallel to $12 - 3y = 4x$. Is there more than one answer? If so, how are the different values for a and b related?

3. Plot points $A = (-6, 1)$, $B = (-4, -3)$, and $C = (2, 0)$. Write equations in a point-slope form for the lines AB and BC. Note that the product of the slopes of these is equal to -1. What can be said about the lines AB and BC looking at the graph?

4. The coordinates of the vertices of quadrilateral $XYZW$ are shown in the diagram below.



   (a) Find a pair of parallel and a pair perpendicular sides in the quadrilateral $XYZW$.
   (b) Write in a point-slope form equations of the lines containing the two parallel sides. How do you know these lines are parallel?
   (c) Write in a point-slope form equations of the lines containing the two perpendicular sides. How do you know these lines are perpendicular?

5. You are in charge of buying the hamburger and boned chicken for a barbecue. The hamburger costs \$2 per pound and the boned chicken costs \$3 per pound. You have \$30 to spend.

   (a) Write an equation that describes the different amounts of hamburger and boned chicken that you can buy.
   (b) Make a table and a scatter plot that illustrate different amounts of hamburger and boned chicken that you can buy.

5.5 Linear inequalities (part 2)

1. Find all the values of x that make $0.1x + 0.25(102 - x) < 17.10$ true.

2. Describe and correct the error.

   (a)

$$\begin{aligned} -4y + 10 &< 15, \\ -4y &< 5, \\ \frac{-4y}{-4} &< \frac{5}{-4}, \\ y &< -\frac{5}{4}. \end{aligned}$$

   (b)

$$\begin{aligned} 6x - 4 &\geq 2x + 1, \\ 6x &\geq 2x - 3, \\ 4x &\geq -3, \\ x &\geq -\frac{3}{4}. \end{aligned}$$

3. Eugene and Wes are solving the inequality $132 - 4x \leq 36$. Each begins by subtracting 132 from both sides to get $-4x \leq -96$, and then each divides both sides by -4. Eugene gets $x \leq 24$ and Wes gets $x \geq 24$, however. Always happy to offer advice, Alex now suggests to Eugene and Wes that answers to inequalities can often be checked by substituting $x = 0$ into both the original inequality and the answer. What do you think of this advice? Graph each of these answers on a number line.

4. (Continuation) After hearing Alex's suggestion about using a test value to check an inequality, Cameron suggests that the problem could have been done by solving the equation $132 - 4x = 36$ first. Complete the reasoning behind this strategy.

5. (Continuation) Deniz, who has been keeping quiet during the discussion, remarks, "The only really tricky thing about inequalities is when you try to multiply them or divide them by negative numbers, but this kind of step can be avoided altogether. Cameron just told us one way to avoid it, and there is another way, too." Explain this remark by Deniz.

5.6 Units (part 2)

1. A car races past you on Front Street, moving at 88 feet per second. This rate is equivalent to how many miles per hour? There are 5280 feet in a mile.

2. Given that Brett can wash d dishes in h hours, write expressions for

   (a) the number of hours it takes for Brett to wash p dishes;

   (b) the number of dishes Brett can wash in y hours;

   (c) the number of dishes Brett can wash in m minutes.

3. If it costs d dollars to buy p gizmos, how much will it cost to buy k gizmos?

4. Four people live in a family. If Diane's scholarship is doubled, the family income will increase by 5%. If mom's salary is doubled, the family income will increase by 30%. If dad's salary is doubled, the family income will increase by 40%. How much will family income increase if Mitchell's research grant is doubled?

5. During 2010, it is estimated that the world consumed 5.20×10^{17} BTUs (British Thermal Units) of energy.

   (a) Describe this estimate of world energy use in *quadrillions* of BTUs. It is now customary to refer to one quadrillion of BTUs as simply a *quad*.

   (b) One barrel of oil produces 5800000 BTUs. How many barrels of oil produce one quad?

   (c) The world is consuming oil at approximately 87 million barrels per day. What is the percentage of world energy consumption attributable to oil?